

Yang-Baxter deformations and string dualities

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Abstract

We further study integrable deformations of the $\text{AdS}_5 \times \text{S}^5$ superstring by following the Yang-Baxter sigma model approach with classical r -matrices satisfying the classical Yang-Baxter equation (CYBE). Deformed string backgrounds specified by r -matrices are considered as solutions of type IIB supergravity, and therefore the relation between gravitational solutions and r -matrices may be called the gravity/CYBE correspondence. In this paper, we present a family of string backgrounds associated with a classical r -matrices carrying two parameters and its three-parameter generalization. The two-parameter case leads to the metric and NS-NS two-form of a solution found by Hubeny-Rangamani-Ross [hep-th/0504034] and another solution in [arXiv:1402.6147]. For all of the backgrounds associated with the three-parameter case, the metric and NS-NS two-form are reproduced by performing TsT transformations and S-dualities for the undeformed $\text{AdS}_5 \times \text{S}^5$ background. As a result, one can anticipate the R-R sector that should be reproduced via a supercoset construction.

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1 Introduction

The AdS/CFT correspondence [1] has a significant property, integrability [2]. It enables us to compute exactly some quantities such as scaling dimensions of composite operators and scattering amplitudes at arbitrary couplings even though those are not protected by supersymmetries. On the string-theory side, the integrability has an intimate relation to classical integrability of non-linear sigma models in two dimensions. In particular, the mathematical structure of the target-space geometry is closely associated with the integrability.

The supergeometry of $\text{AdS}_5 \times \text{S}^5$ is represented by the following supercoset,

$$\frac{PSU(2, 2|4)}{SO(1, 4) \times SO(5)} . \quad (1.1)$$

The Green-Schwarz type action is constructed based on this supercoset [3]. It is known that the coset (1.1) enjoys the \mathbb{Z}_4 -grading property and it ensures the classical integrability [4]. For another formulation [5], the classical integrability is discussed in [6].

What kinds of string backgrounds would be concerned with integrability? It is well-known that symmetric cosets in general lead to classical integrability. For the symmetric cosets, consistent string backgrounds have been classified in [7, 8]. On the other hand, some of non-symmetric cosets may lead to integrable sigma models, but there is no general prescription. Thus one has to study case by case and hence the main focus has been on simple cases. In the case of squashed S^3 , a q -deformation of $\mathfrak{su}(2)$ and its affine extension have been shown in [9–11]. Similar results have also been obtained for 3D Schrödinger spacetimes [12, 13]. For an earlier attempt to study higher-dimensional cases, see [14]. For integrable deformations of Wess-Zumino-Novikov-Witten models, see [15–18].

A systematic method to study integrable deformations was proposed by Klimcik [19]. The original form was applicable only to principal chiral models and it was based on classical r -matrices satisfying modified classical Yang-Baxter equation (mCYBE). Then it was generalized to symmetric coset cases [20]. Just after that, it was further applied to a q -deformation of the $\text{AdS}_5 \times S^5$ superstring [21]. The q -deformed metric (in the string frame) and the NS-NS two-form were derived in [22]. Notably, the metric exhibits a singularity surface. A generalization to other cases are studied in [23]. A mirror description has been proposed in [24, 25]. A possible resolution of the singularity has been argued by taking the fast-moving string limit [26]. Giant magnon solutions are studied in [24, 27]. Deformed Neumann models are constructed [28]. A new coordinate system has been argued in [29]. Although it seems quite difficult to fix the dilaton and the R-R sector for the q -deformed $\text{AdS}_5 \times S^5$ superstring, the dilaton was determined at least in lower-dimensional cases such as $\text{AdS}_2 \times S^2$ [30]. For two-parameter generalizations, see [23, 31].

It is also possible to consider another kind of integrable deformations of the $\text{AdS}_5 \times S^5$ superstring based on the classical Yang-Baxter equation (CYBE) [32], rather than mCYBE. We have already found out some classical r -matrices, which correspond to solutions of type IIB supergravity. An example is the Lunin-Maldacena-Frolov backgrounds [33, 34] and the corresponding r -matrix is composed of the Cartan generators of $\mathfrak{su}(4)$ [35]. Another example is gravity duals for non-commutative gauge theories [36–39]. The associated r -matrices are of peculiar Jordanian type [40]. A rather remarkable example is the $\text{AdS}_5 \times T^{1,1}$ case. The $T^{1,1}$ background is argued to be non-integrable [41]. On the other hand, the $T^{1,1}$ geometry can be represented by a coset and hence one may consider Yang-Baxter deformations of $T^{1,1}$. Then

it has been shown in [42] that the resulting geometry nicely agrees with γ -deformations of $T^{1,1}$ previously discussed in [33, 43]. For a short summary of the works based on the CYBE, see [44].

In this paper, we further study integrable deformations of the $\text{AdS}_5 \times \text{S}^5$ superstring along the line of [32]. Deformed string backgrounds specified by r -matrices are considered as solutions of type IIB supergravity, and therefore the relation between gravitational solutions and r -matrices may be called the gravity/CYBE correspondence. We will present here a family of string backgrounds associated with a classical r -matrices carrying two parameters and its three-parameter generalization. The two-parameter case leads to the metric and NS-NS B-field of a solution found by Hubeny-Rangamani-Ross [45] and another solution in [46] as special cases. More generally, for all of the backgrounds associated with the three-parameter case, the metric and NS-NS two-form are reproduced by performing TsT transformations and S-dualities for the undeformed $\text{AdS}_5 \times \text{S}^5$ background. As a result, one can anticipate the R-R sector that should be reproduced via a supercoset construction.

A remarkable point is that the solution obtained in [46] has been reproduced by performing a duality chain for the $\text{AdS}_5 \times \text{S}^5$, and as a result it is shown to be a consistent string background. Namely, it is automatically ensured that the beta function vanishes. This point should be stressed against those who are skeptical about the solution in [46].

This paper is organized as follows. Section 2 provides a short review of Yang-Baxter deformations of the $\text{AdS}_5 \times \text{S}^5$ superstring. We present a classical r -matrix with two parameters, which is a solution of the CYBE. Then the deformed metric and NS-NS two-form are obtained by evaluating the classical action. In section 3, we reproduce the resulting metric and NS-NS two-form by performing a chain of dualities for the $\text{AdS}_5 \times \text{S}^5$ background. Section 4 generalizes the previous argument to a three-parameter case and considers the correspondence between the deformations of $\text{AdS}_5 \times \text{S}^5$ and the associated classical r -matrices. Section 5 is devoted to conclusion and discussion.

In Appendix A, our notation and convention are summarized. The Buscher rules of T-duality are listed in Appendix B.

2 Jordanian deformations of the $\text{AdS}_5 \times \text{S}^5$ superstring

In subsection 2.1, let us first recall the formulation of Yang-Baxter sigma models. Then we consider a two-parameter deformation of the AdS_5 in subsection 2.2. The resulting metric and NS-NS two-form are computed in subsection 2.3.

2.1 Yang-Baxter deformations of the $\text{AdS}_5 \times \text{S}^5$ superstring

The Yang-Baxter sigma model approach [19–21] is applicable to the $\text{AdS}_5 \times \text{S}^5$ superstring by using a classical r -matrix satisfying the CYBE [32]. Then the deformed action is given by

$$S = -\frac{1}{4}(\gamma^{\alpha\beta} - \epsilon^{\alpha\beta}) \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \text{STr} \left(A_\alpha d \circ \frac{1}{1 - \eta R_g \circ d} A_\beta \right), \quad (2.1)$$

where the left-invariant one-form is defined as

$$A_\alpha \equiv g^{-1} \partial_\alpha g, \quad g \in SU(2, 2|4). \quad (2.2)$$

By taking the parameter $\eta \rightarrow 0$, the action (2.1) reduces to the undeformed one [3]. Here the flat metric $\gamma^{\alpha\beta}$ and the anti-symmetric tensor $\epsilon^{\alpha\beta}$ on the string world-sheet are normalized as $\gamma^{\alpha\beta} = \text{diag}(-1, 1)$ and $\epsilon^{\tau\sigma} = 1$, respectively. The operator R_g is defined as

$$R_g(X) \equiv g^{-1} R(gXg^{-1})g, \quad (2.3)$$

where a linear operator R is a solution of CYBE rather than mCYBE. The R -operator is related to a classical r -matrix in the tensorial notation through

$$R(X) = \text{STr}_2[r(1 \otimes X)] = \sum_i (a_i \text{STr}(b_i X) - b_i \text{STr}(a_i X)) \quad (2.4)$$

with $r = \sum_i a_i \wedge b_i \equiv \sum_i (a_i \otimes b_i - b_i \otimes a_i).$

The generators a_i, b_i are some elements of $\mathfrak{su}(2, 2|4)$. The operator d is defined by

$$d \equiv P_1 + 2P_2 - P_3, \quad (2.5)$$

with projectors P_k ($k = 0, 1, 2, 3$) from $\mathfrak{su}(2, 2|4)$ to its \mathbb{Z}_4 -graded components $\mathfrak{su}(2, 2|4)^{(k)}$. In particular, $\mathfrak{su}(2, 2|4)^{(0)}$ is a gauge symmetry, $\mathfrak{so}(1, 4) \oplus \mathfrak{so}(5)$.

2.2 A two-parameter deformation of AdS_5

We consider here a deformation of the AdS_5 bosonic part of (2.1) based on the following classical r -matrix of Jordanian type,

$$r_{\text{Jor}} = E_{24} \wedge (c_1 E_{22} - c_2 E_{44}), \quad (2.6)$$

where E_{ij} ($i, j = 1, 2, 3, 4$) are the fundamental representation of the generators of $\mathfrak{su}(2, 2)$ defined by $(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$. Here c_1 and c_2 are constant complex numbers.

To evaluate the action, it is convenient to rewrite the metric part and NS-NS two-form coupled part of the Lagrangian (2.1) into the following form,

$$\begin{aligned} L_G &= \frac{1}{2} [A_\tau P_2(J_\tau) - A_\sigma P_2(J_\sigma)] , \\ L_B &= \frac{1}{2} [A_\tau P_2(J_\sigma) - A_\sigma P_2(J_\tau)] , \end{aligned} \quad (2.7)$$

where J_α is a projected current defined as

$$J_\alpha \equiv \frac{1}{1 - 2[R_{\text{Jor}}]_g \circ P_2} A_\alpha . \quad (2.8)$$

Here the parameter η in (2.1) is set as $\eta = 1$. The linear R-operator R_{Jor} associated with (2.6) is represented by the identification (2.4).

To find the AdS_5 part of (2.7), we use the following parameterization,

$$g = \exp(p_0 x^0 + p_1 x^1 + p_2 x^2 + p_3 x^3) \exp\left(\frac{\gamma_5}{2} \log z\right) \in SU(2, 2) . \quad (2.9)$$

For the $\mathfrak{su}(2, 2)$ generators p_μ ($\mu = 0, 1, 2, 3$) and γ_5 , see Appendix A. The projected deformed current $P_2(J_\alpha)$ is obtained by solving the equation,

$$(1 - 2P_2 \circ R_g)P_2(J_\alpha) = P_2(A_\alpha) . \quad (2.10)$$

By plugging the projected current¹

$$P_2(A_\alpha) = \frac{\partial_\alpha x^0 \gamma_0 + \partial_\alpha x^1 \gamma_1 + \partial_\alpha x^2 \gamma_2 + \partial_\alpha x^3 \gamma_3 + \partial_\alpha z \gamma_5}{2z} \quad (2.11)$$

with (2.10), the deformed current is evaluated as

$$P_2(J_\alpha) = j_\alpha^0 \gamma_0 + j_\alpha^1 \gamma_1 + j_\alpha^2 \gamma_2 + j_\alpha^3 \gamma_3 + j_\alpha^5 \gamma_5 , \quad (2.12)$$

with the coefficients

$$\begin{aligned} j_\alpha^1 &= \frac{1}{2z} \partial_\alpha x^1 - \frac{(c_1 + c_2)x^1 + i(c_1 - c_2)x^2}{2\sqrt{2}z^3} \partial_\alpha x^+ , \\ j_\alpha^2 &= \frac{1}{2z} \partial_\alpha x^2 - \frac{(c_1 + c_2)x^2 - i(c_1 - c_2)x^1}{2\sqrt{2}z^3} \partial_\alpha x^+ , \\ j_\alpha^0 + j_\alpha^3 &= \frac{1}{\sqrt{2}z} \partial_\alpha x^+ , \\ j_\alpha^0 - j_\alpha^3 &= -\frac{(c_1 + c_2)x^1 + i(c_1 - c_2)x^2}{2z^3} \partial_\alpha x^1 - \frac{(c_1 + c_2)x^2 - i(c_1 - c_2)x^1}{2z^3} \partial_\alpha x^2 \end{aligned}$$

¹For the convention of γ -matrices, see Appendix A.

$$\begin{aligned}
& + \frac{4c_1c_2((x^1)^2 + (x^2)^2) + (c_1 + c_2)^2z^2}{2\sqrt{2}z^5} \partial_\alpha x^+ + \frac{1}{\sqrt{2}z} \partial_\alpha x^- - \frac{c_1 + c_2}{2z^2} \partial_\alpha z, \\
j_\alpha^5 &= \frac{1}{2z} \partial_\alpha z - \frac{c_1 + c_2}{2\sqrt{2}z^2} \partial_\alpha x^+.
\end{aligned} \tag{2.13}$$

Here we have introduced the light-cone coordinates

$$x^\pm \equiv \frac{x^0 \pm x^3}{\sqrt{2}}. \tag{2.14}$$

Finally, the metric part and NS-NS two-form part of the Lagrangian (2.7) are given by

$$\begin{aligned}
L_G &= -\gamma^{\alpha\beta} \left[\frac{-2\partial_\alpha x^+ \partial_\beta x^- + \partial_\alpha x^1 \partial_\beta x^1 + \partial_\alpha x^2 \partial_\beta x^2 + \partial_\alpha z \partial_\beta z}{2z^2} \right. \\
&\quad \left. - \frac{4c_1c_2((x^1)^2 + (x^2)^2) + (c_1 + c_2)^2z^2}{8z^6} \partial_\alpha x^+ \partial_\beta x^+ \right], \\
L_B &= -\epsilon^{\alpha\beta} \left[(c_1 + c_2) \frac{(x^1 \partial_\alpha x^1 + x^2 \partial_\alpha x^2 + z \partial_\alpha z) \partial_\beta x^+}{2z^4} + i(c_1 - c_2) \frac{(x^2 \partial_\alpha x^1 - x^1 \partial_\alpha x^2) \partial_\beta x^+}{2z^4} \right].
\end{aligned} \tag{2.15}$$

Note that the resulting Lagrangian becomes complex in general. Therefore it is necessary to argue the reality condition.

Reality condition

Interestingly, both L_G and L_B become real if and only if c_1 and c_2 are related by the complex conjugation,

$$c_1 = c_2^* \iff c_1 = \alpha, \quad c_2 = \alpha^* \quad (\alpha \in \mathbb{C}). \tag{2.16}$$

In particular, the result given in Subsec. 2.2 of [46] can be reproduced by imposing that

$$c_1 = c_2 = \frac{1}{\sqrt{2}}. \tag{2.17}$$

2.3 The metric and NS-NS two-form

In the previous subsection, we have seen that a Jordanian r -matrix (2.6) yields the metric in the string frame and NS-NS two form, as presented in (2.15). Taking into account the reality condition (2.16), the associated metric and NS-NS two form are derived as

$$\begin{aligned}
ds^2 &= \frac{-2dx^+ dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \frac{|\alpha|^2((x^1)^2 + (x^2)^2) + (\text{Re}(\alpha))^2 z^2}{z^6} (dx^+)^2, \\
B_2 &= -\text{Re}(\alpha) \frac{(x^1 dx^1 + x^2 dx^2 + z dz) \wedge dx^+}{z^4} + \text{Im}(\alpha) \frac{(x^2 dx^1 - x^1 dx^2) \wedge dx^+}{z^4}.
\end{aligned} \tag{2.18}$$

It is worth seeing two special cases of α as listed below.

(i) Pure imaginary α -deformation

When α is pure imaginary,

$$\alpha \equiv i\alpha_I \quad \text{with} \quad \alpha_I \in \mathbb{R}, \quad (2.19)$$

the metric and NS-NS two-form in (2.18) are given by

$$\begin{aligned} ds^2 &= \frac{-2dx^+dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \alpha_I^2 \frac{(x^1)^2 + (x^2)^2}{z^6} (dx^+)^2, \\ B_2 &= \alpha_I \frac{(x^2 dx^1 - x^1 dx^2) \wedge dx^+}{z^4}. \end{aligned} \quad (2.20)$$

These agree with the ones of the solution found in [45].

(ii) Real α -deformation

When α is a real number,

$$\alpha = \alpha_R \quad \text{with} \quad \alpha_R \in \mathbb{R}, \quad (2.21)$$

the metric and NS-NS two-form in (2.18) is rewritten as

$$\begin{aligned} ds^2 &= \frac{-2dx^+dx^- + (dx^1)^2 + (dx^2)^2 + dz^2}{z^2} - \alpha_R^2 \frac{(x^1)^2 + (x^2)^2 + z^2}{z^6} (dx^+)^2, \\ B_2 &= -\alpha_R \frac{(x^1 dx^1 + x^2 dx^2 + z dz) \wedge dx^+}{z^4}. \end{aligned} \quad (2.22)$$

The above metric and NS-NS two-form are nothing but the ones obtained in [46].

3 A chain of dualities for $\text{AdS}_5 \times \text{S}^5$

In this section, we show that the metric and NS-NS two-form in (2.18) are reproduced by performing TsT-transformations and an S-duality for the $\text{AdS}_5 \times \text{S}^5$ background.

In order to perform T-dualities, let us introduce the polar coordinates on the x^1 - x^2 plane,

$$x^1 = \rho \cos \varphi, \quad x^2 = \rho \sin \varphi \quad (0 \leq \rho < \infty, \quad 0 \leq \varphi < 2\pi). \quad (3.1)$$

With these coordinates, the original $\text{AdS}_5 \times \text{S}^5$ background is given by

$$\begin{aligned} ds^2 &= \frac{-2dx^+dx^- + d\rho^2 + \rho^2 d\varphi^2 + dz^2}{z^2} + (d\chi + \omega)^2 + ds_{\text{CP}^2}^2, \\ F_5 &= dC_4 = 4 \left[-\frac{1}{z^5} dx^+ \wedge dx^- \wedge d\rho \wedge \rho d\varphi \wedge dz \right. \end{aligned} \quad (3.2)$$

$$\begin{aligned}
& + (d\chi + \omega) \wedge d\mu \wedge \sin \mu \Sigma_1 \wedge \sin \mu \Sigma_2 \wedge \cos \mu \sin \mu \Sigma_3 \Big] , \\
& B_2 = C_2 = C = 0 , \quad \Phi = \Phi_0 \text{ (const.)} .
\end{aligned}$$

Note that the metric of S^5 is expressed as a $U(1)$ -fibration over \mathbb{CP}^2 . Here χ is a local coordinate on the fiber and ω is a one-form potential for the Kähler form on \mathbb{CP}^2 . In the usual way, the metric of \mathbb{CP}^2 and ω are given by

$$ds_{\mathbb{CP}^2}^2 = d\mu^2 + \sin^2 \mu (\Sigma_1^2 + \Sigma_2^2 + \cos^2 \mu \Sigma_3^2) , \quad \omega \equiv \sin^2 \mu \Sigma_3 , \quad (3.3)$$

where Σ_a ($a = 1, 2, 3$) are defined as

$$\begin{aligned}
\Sigma_1 &\equiv \frac{1}{2} (\cos \psi d\theta + \sin \psi \sin \theta d\phi) , & \Sigma_2 &\equiv \frac{1}{2} (\sin \psi d\theta - \cos \psi \sin \theta d\phi) , \\
\Sigma_3 &\equiv \frac{1}{2} (d\psi + \cos \theta d\phi) .
\end{aligned}$$

The geometry of S^5 is described with the five coordinates: $(\chi, \mu, \psi, \theta, \phi)$.

For later computations, it is nice to write down explicitly the R-R four-form C_4 ,

$$C_4 = \frac{1}{z^4} dx^+ \wedge dx^- \wedge d\rho \wedge \rho d\varphi - \sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3 . \quad (3.4)$$

Note that the following relations are satisfied,

$$\Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3 = -\frac{1}{8} \sin \theta d\theta \wedge d\phi \wedge d\psi , \quad d(\Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3) = 0 .$$

It is also helpful to use the relations,

$$dx^1 \wedge dx^2 = d\rho \wedge \rho d\varphi , \quad x^1 dx^1 + x^2 dx^2 = \rho d\rho , \quad x^2 dx^1 - x^1 dx^2 = -\rho^2 d\varphi .$$

In the following, we will apply TsT transformations and an S-duality for the $\text{AdS}_5 \times S^5$ background in (3.2).

3.1 The first TsT transformation

The first step is to perform a TsT transformation to the $\text{AdS}_5 \times S^5$ background in (3.2).

Let us first take a T-duality along the φ -direction in (3.2). According to the rules of T-duality (listed in Appendix B), the background is rewritten as

$$\begin{aligned}
d\tilde{s}^2 &= \frac{-2dx^+ dx^- + d\rho^2 + dz^2}{z^2} + \frac{z^2}{\rho^2} d\tilde{\varphi}^2 + ds_{S^5}^2 , \\
\tilde{B}_2 &= 0 , \quad \tilde{\Phi} = \Phi_0 - \frac{1}{2} \ln \left(\frac{\rho^2}{z^2} \right) ,
\end{aligned}$$

$$\tilde{C}_3 = -\frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho, \quad \tilde{C}_5 = -\sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3 \wedge d\tilde{\varphi}. \quad (3.5)$$

Then the x^- -coordinate is shifted as

$$x^- \rightarrow x^- + a_1 \tilde{\varphi} \quad (3.6)$$

with a real parameter a_1 . The metric and \tilde{C}_3 are deformed as follows:

$$\begin{aligned} d\tilde{s}^2 &= \frac{-2dx^+ dx^- + d\rho^2 + dz^2}{z^2} - 2\frac{a_1}{z^2} dx^+ d\tilde{\varphi} + \frac{z^2}{\rho^2} d\tilde{\varphi}^2 + ds_{S^5}^2, \\ \tilde{B}_2 &= 0, \quad \tilde{\Phi} = \Phi_0 - \frac{1}{2} \ln \left(\frac{\rho^2}{z^2} \right), \\ \tilde{C}_3 &= -\frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho - a_1 \frac{\rho}{z^4} dx^+ \wedge d\tilde{\varphi} \wedge d\rho, \\ \tilde{C}_5 &= -\sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3 \wedge d\tilde{\varphi}. \end{aligned} \quad (3.7)$$

Finally, by taking a T-duality along the $\tilde{\varphi}$ -direction, the resulting background is given by

$$\begin{aligned} ds^2 &= \frac{-2dx^+ dx^- + d\rho^2 + \rho^2 d\varphi^2 + dz^2}{z^2} - a_1^2 \frac{\rho^2}{z^6} (dx^+)^2 + ds_{S^5}^2, \\ B_2 &= -a_1 \frac{\rho^2}{z^4} dx^+ \wedge d\varphi, \quad \Phi = \Phi_0, \quad C_2 = a_1 \frac{\rho}{z^4} dx^+ \wedge d\rho, \\ C_4 &= \frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho \wedge d\varphi - \sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3. \end{aligned} \quad (3.8)$$

Note that the dilaton is constant. Through the coordinate transformations (3.1), the above metric and NS-NS two-form agree with the ones in (2.20) when $a_1 = -\text{Im}(\alpha) = -\alpha_I$. The NS-NS sector of the background (3.8) has already been obtained in [45].

3.2 The second TsT transformation

The next step is to perform another TsT transformation for the background (3.8). This process is essentially the same as the one in [47].

Let us first take a T-duality along the χ -direction of S^5 . The resulting metric is given by

$$\begin{aligned} ds^2 &= \frac{-2dx^+ dx^- + d\rho^2 + \rho^2 d\varphi^2 + dz^2}{z^2} - a_1^2 \frac{\rho^2}{z^6} (dx^+)^2 + d\tilde{\chi}^2 + ds_{\mathbb{CP}^2}^2, \\ \tilde{B}_2 &= -a_1 \frac{\rho^2}{z^4} dx^+ \wedge d\varphi + \frac{1}{2} \sin^2 \mu d\psi \wedge d\tilde{\chi} + \frac{1}{2} \sin^2 \mu \cos \theta d\phi \wedge d\tilde{\chi}, \quad \Phi = \Phi_0, \\ \tilde{C}_3 &= a_1 \frac{\rho}{z^4} dx^+ \wedge d\rho \wedge d\tilde{\chi} - \sin^4 \mu \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3, \\ \tilde{C}_5 &= \frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho \wedge d\varphi \wedge d\tilde{\chi}. \end{aligned} \quad (3.9)$$

Then, by shifting x^- as

$$x^- \rightarrow x^- + a_2 \tilde{\chi}, \quad (3.10)$$

only the metric is deformed as

$$ds^2 = \frac{-2dx^+ dx^- + d\rho^2 + \rho^2 d\varphi^2 + dz^2}{z^2} - a_1^2 \frac{\rho^2}{z^6} (dx^+)^2 - 2 \frac{a_2}{z^2} dx^+ d\tilde{\chi} + d\tilde{\chi}^2 + ds_{\mathbb{CP}^2}^2.$$

Finally, by taking a T-duality along the $\tilde{\chi}$ -direction, the resulting background is given by

$$\begin{aligned} ds^2 &= \frac{-2dx^+ dx^- + d\rho^2 + \rho^2 d\varphi^2 + dz^2}{z^2} - \left(a_1^2 \frac{\rho^2}{z^6} + a_2^2 \frac{1}{z^4} \right) (dx^+)^2 + (d\chi + \omega)^2 + ds_{\mathbb{CP}^2}^2, \\ B_2 &= -a_1 \frac{\rho^2}{z^4} dx^+ \wedge d\varphi - \frac{a_2}{z^2} dx^+ \wedge (d\chi + \omega), \quad \Phi = \Phi_0, \quad C_2 = a_1 \frac{\rho}{z^4} dx^+ \wedge d\rho, \\ C_4 &= \frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho \wedge d\varphi - \sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3. \end{aligned} \quad (3.11)$$

As a result, a deformation term of the Schrödinger spacetime has been added.

When $a_1 = 0$, the Schrödinger solution is realized [47]. In comparison to the r -matrix (2.6) that is composed of $\mathfrak{su}(2, 2)$ only, the classical r -matrix associated with this solution itself is composed of both $\mathfrak{su}(2, 2)$ and $\mathfrak{su}(4)$ generators because the TsT transformation includes a direction of S^5 . We will explain in very detail the r -matrix corresponding to the Schrödinger spacetime in another place [48]. Notably, this result indicates that the Schrödinger spacetime is integrable². A further remarkable point is that brane-wave type deformations [51] also lead to integrable backgrounds.

3.3 S-duality

Then let us perform an S-duality for the background (3.11).

The transformation rule of S-duality is given by

$$\Phi' = -\Phi, \quad B'_2 = C_2, \quad C'_2 = -B_2. \quad (3.12)$$

Hence, after performing the S-duality, the resulting background is given by

$$\begin{aligned} ds^2 &= \frac{-2dx^+ dx^- + d\rho^2 + \rho^2 d\varphi^2 + dz^2}{z^2} - \left(a_1^2 \frac{\rho^2}{z^6} + a_2^2 \frac{1}{z^4} \right) (dx^+)^2 + (d\chi + \omega)^2 + ds_{\mathbb{CP}^2}^2, \\ B_2 &= a_1 \frac{\rho}{z^4} dx^+ \wedge d\rho, \quad \Phi = -\Phi_0, \quad C_2 = a_1 \frac{\rho^2}{z^4} dx^+ \wedge d\varphi + \frac{a_2}{z^2} dx^+ \wedge (d\chi + \omega), \end{aligned}$$

² The integrability of Schrödinger spacetimes may be related to the coset structure argued in [49]. A classification of super Schrödinger algebras [50] would also play an important role in the future.

$$C_4 = \frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho \wedge d\varphi - \sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3. \quad (3.13)$$

When $a_1 = \pm a_2 = \text{Re}(\alpha) = \alpha_R$, the above background agrees with the one in (2.22). This solution is nothing but the one obtained in [46], where the deformation parameter was denoted by $\alpha_R \equiv \eta$. Notably, this result ensures that the solution in [46] is a consistent string background. In particular, the world-sheet beta-function vanishes.

3.4 The third TsT transformation

Furthermore, let us perform a TsT-transformation, which is the same as in Sec. 3.1, for the background in (3.13).

The background in (3.13) seems complicated, but it has a $U(1)$ symmetry along the φ -direction. We first take a T-duality along the φ -direction. As a result, the background is transformed as

$$\begin{aligned} ds^2 &= \frac{-2dx^+ dx^- + d\rho^2 + dz^2}{z^2} + \frac{z^2}{\rho^2} d\tilde{\varphi}^2 - \left(a_1^2 \frac{\rho^2}{z^6} + a_2^2 \frac{1}{z^4} \right) (dx^+)^2 + (d\chi + \omega)^2 + ds_{\mathbb{CP}^2}^2, \\ B_2 &= a_1 \frac{\rho}{z^4} dx^+ \wedge d\rho, \quad \Phi = -\Phi_0 - \frac{1}{2} \ln \left(\frac{\rho^2}{z^2} \right), \quad C_1 = -a_1 \frac{\rho^2}{z^4} dx^+, \\ C_3 &= \frac{a_2}{z^2} dx^+ \wedge (d\chi + \omega) \wedge d\tilde{\varphi} - \frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho, \\ C_5 &= -\sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3 \wedge d\tilde{\varphi}. \end{aligned} \quad (3.14)$$

Then, by shifting x^- as

$$x^- \rightarrow x^- + a_3 \tilde{\varphi}, \quad (3.15)$$

it is rewritten into the following form:

$$\begin{aligned} ds^2 &= \frac{-2dx^+ dx^- + d\rho^2 + dz^2}{z^2} - 2\frac{a_3}{z^2} dx^+ d\tilde{\varphi} + \frac{z^2}{\rho^2} d\tilde{\varphi}^2 - \left(a_1^2 \frac{\rho^2}{z^6} + a_2^2 \frac{1}{z^4} \right) (dx^+)^2 \\ &\quad + (d\chi + \omega)^2 + ds_{\mathbb{CP}^2}^2, \\ B_2 &= a_1 \frac{\rho}{z^4} dx^+ \wedge d\rho, \quad \Phi = -\Phi_0 - \frac{1}{2} \ln \left(\frac{\rho^2}{z^2} \right), \quad C_1 = -a_1 \frac{\rho^2}{z^4} dx^+, \\ C_3 &= \frac{a_2}{z^2} dx^+ \wedge (d\chi + \omega) \wedge d\tilde{\varphi} - \frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho + a_3 \frac{\rho}{z^4} dx^+ \wedge d\rho \wedge d\tilde{\varphi}, \\ C_5 &= -\sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3 \wedge d\tilde{\varphi}. \end{aligned} \quad (3.16)$$

Finally, by taking a T-duality along the $\tilde{\varphi}$ -direction, the resulting background is given by

$$ds^2 = \frac{-2dx^+ dx^- + d\rho^2 + \rho^2 d\varphi^2 + dz^2}{z^2} - \left((a_1^2 + a_3^2) \frac{\rho^2}{z^6} + a_2^2 \frac{1}{z^4} \right) (dx^+)^2$$

$$\begin{aligned}
& + (d\chi + \omega)^2 + ds_{\mathbb{CP}^2}^2, \\
B_2 &= a_1 \frac{\rho}{z^4} dx^+ \wedge d\rho - a_3 \frac{\rho^2}{z^4} dx^+ \wedge d\varphi, \quad \Phi = -\Phi_0, \\
C_2 &= a_1 \frac{\rho^2}{z^4} dx^+ \wedge d\varphi + \frac{a_2}{z^2} dx^+ \wedge (d\chi + \omega) + a_3 \frac{\rho}{z^4} dx^+ \wedge d\rho, \\
C_4 &= \frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho \wedge d\varphi - \sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3.
\end{aligned} \tag{3.17}$$

When $a_1 = \pm a_2 = \text{Re}(\alpha)$ and $a_3 = -\text{Im}(\alpha)$, the solution (3.17) reproduces the metric and NS-NS two-form in (2.18). Note that in this identification the sign concerning a_2 has not been determined here. In order to fix this ambiguity, we have to perform a supercoset construction.

So far, the R-R sector has not been determined yet from the Yang-Baxter sigma model approach. But the solution (3.17) gives a prediction for the R-R sector. If the gravity/CYBE correspondence is true, then the R-R sector of (3.17) should be reproduced by performing a supercoset construction. We will not try to do that here and leave it as a future problem. However, it should be derived as expected. The kappa-invariance of the deformed string action would also imply the agreement.

4 Duality-chains and classical r -matrices

It is worth summarizing the relation between duality-chains and classical r -matrices. This is nothing but the gravity/CYBE correspondence concerned with the present scope.

For the completeness, we first consider the fourth TsT-transformation in subsection 4.1 and the second S-duality in subsection 4.2. In subsection 4.3, we propose a classical r -matrix including three parameters and summarize a concrete realization of the gravity/CYBE correspondence.

4.1 The fourth TsT transformation

Let us consider here a TsT-transformation for the background in (3.17).

We first perform a T-duality for the background in (3.17) along the χ -direction. The resulting background is

$$\begin{aligned}
ds^2 &= ds_{\text{AdS}_5}^2 - \left((a_1^2 + a_3^2) \frac{\rho^2}{z^6} + a_2^2 \frac{1}{z^4} \right) (dx^+)^2 + d\tilde{\chi}^2 + ds_{\mathbb{CP}^2}^2, \\
B_2 &= dx^+ \wedge \left(a_1 \frac{\rho d\rho}{z^4} - a_3 \frac{\rho^2 d\varphi}{z^4} \right) + \omega \wedge d\tilde{\chi}, \quad \tilde{\Phi} = -\Phi_0, \quad C_1 = -\frac{a_2}{z^2} dx^+,
\end{aligned}$$

$$\begin{aligned}
C_3 &= dx^+ \wedge \left(a_1 \frac{\rho^2 d\varphi}{z^4} + a_3 \frac{\rho d\rho}{z^4} \right) \wedge \tilde{d}\chi - \sin^4 \mu \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3, \\
C_5 &= \frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho \wedge d\phi \wedge d\tilde{\chi},
\end{aligned} \tag{4.1}$$

where the undeformed metric of AdS_5 is denoted as

$$ds_{\text{AdS}_5}^2 = \frac{-2dx^+ dx^- + d\rho^2 + \rho^2 d\varphi^2 + dz^2}{z^2}. \tag{4.2}$$

Then, by shifting the x^- -coordinate as

$$x^- \rightarrow x^- + a_4 \tilde{\chi}, \tag{4.3}$$

only the metric in (4.1) is modified as

$$ds^2 = ds_{\text{AdS}_5}^2 - \frac{2a_4}{z^2} dx^+ d\tilde{\chi} - \left((a_1^2 + a_3^2) \frac{\rho^2}{z^6} + a_2^2 \frac{1}{z^4} \right) (dx^+)^2 + d\tilde{\chi}^2 + ds_{\text{CP}^2}^2. \tag{4.4}$$

Finally, by T-dualizing back along the $\tilde{\chi}$ -direction, we arrive at the following background:

$$\begin{aligned}
ds^2 &= ds_{\text{AdS}_5}^2 - \left((a_1^2 + a_3^2) \frac{\rho^2}{z^6} + (a_2^2 + a_4^2) \frac{1}{z^4} \right) (dx^+)^2 + (d\chi + \omega)^2 + ds_{\text{CP}^2}^2, \\
B_2 &= dx^+ \wedge \left(-a_3 \frac{\rho^2 d\varphi}{z^4} - a_4 \frac{d\chi + \omega}{z^2} + a_1 \frac{\rho d\rho}{z^4} \right), \quad \Phi = -\Phi_0, \\
C_2 &= dx^+ \wedge \left(a_1 \frac{\rho^2 d\varphi}{z^4} + a_2 \frac{d\chi + \omega}{z^2} + a_3 \frac{\rho d\rho}{z^4} \right), \\
C_4 &= \frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho \wedge d\varphi - \sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3.
\end{aligned} \tag{4.5}$$

4.2 The second S-duality

One may notice that, in the background (4.5), the NS-NS two-form B_2 and the Ramond-Ramond two-form C_2 are turned on in a symmetric way. This is because the background (4.5) is obtained by two same TsT-transformations before and after the S-duality (3.12). Then, let us consider the second S-duality of (4.5). The resulting background is given by

$$\begin{aligned}
ds^2 &= ds_{\text{AdS}_5}^2 - \left((a_1^2 + a_3^2) \frac{\rho^2}{z^6} + (a_2^2 + a_4^2) \frac{1}{z^4} \right) (dx^+)^2 + (d\chi + \omega)^2 + ds_{\text{CP}^2}^2, \\
B_2 &= dx^+ \wedge \left(a_1 \frac{\rho^2 d\varphi}{z^4} + a_2 \frac{d\chi + \omega}{z^2} + a_3 \frac{\rho d\rho}{z^4} \right), \quad \Phi = \Phi_0, \\
C_2 &= dx^+ \wedge \left(a_3 \frac{\rho^2 d\varphi}{z^4} + a_4 \frac{d\chi + \omega}{z^2} - a_1 \frac{\rho d\rho}{z^4} \right), \\
C_4 &= \frac{\rho}{z^4} dx^+ \wedge dx^- \wedge d\rho \wedge d\varphi - \sin^4 \mu d\chi \wedge \Sigma_1 \wedge \Sigma_2 \wedge \Sigma_3.
\end{aligned} \tag{4.6}$$

4.3 A three-parameter deformation of $\text{AdS}_5 \times \text{S}^5$

Now it is turn to consider a classical r -matrix corresponding to the background (4.6). It is argued to be the following form,

$$r_{12}(b_1, b_2, b_3) = \frac{1}{\sqrt{2}i} E_{24} \wedge \left(b_1(E_{22} + E_{44}) + \frac{b_2}{2}(h_4 + h_5 + h_6) + ib_3(E_{22} - E_{44}) \right). \quad (4.7)$$

Here h_4, h_5 and h_6 are the three Cartan generators³ of $\mathfrak{su}(4)$ rather than $\mathfrak{su}(2, 2)$, and b_1, b_2, b_3 are real deformation parameters. A direct computation shows that this is a solution of the CYBE.

Plugging the classical r -matrix (4.7) with the classical action (2.1), the resulting metric and NS-NS two-form turn out to be

$$ds^2 = ds_{\text{AdS}_5}^2 + ds_{\text{S}^5}^2 - \frac{(b_1^2 + b_3^2)\rho^2 + (b_2^2 + b_3^2)z^2}{z^6} (dx^+)^2, \quad (4.8)$$

$$B_2 = dx^+ \wedge \left(-b_1 \frac{\rho^2 d\varphi}{z^4} - b_2 \frac{d\chi + \omega}{z^2} + b_3 \frac{\rho d\rho}{z^4} \right), \quad (4.9)$$

where $ds_{\text{AdS}_5}^2$ and $ds_{\text{S}^5}^2$ are the undeformed metrics of AdS_5 and S^5 , respectively.

Indeed, the above metric and NS-NS two-form agree with the ones obtained by the fourth TsT-transformation in (4.5) and the S-duality in (4.6) with the following parameter identifications, respectively,

$$b_1 = a_3, \quad b_2 = a_4, \quad b_3 = a_1 = \pm a_2 \quad \text{for} \quad (4.5), \quad (4.10)$$

$$b_1 = -a_1, \quad b_2 = -a_2, \quad b_3 = a_3 = \pm a_4 \quad \text{for} \quad (4.6). \quad (4.11)$$

It is worth comparing the above results with the deformed backgrounds from TsT-transformations and an S-duality in Sec. 3. The comprehensive relations are summarized in Tab. 1. Here the symbol $(\text{TsT})_\varphi^{a_1}$, for instance, stands for a TsT-transformation consisting of a T-duality for the φ -direction and the shift $x^- \rightarrow x^- + a_1 \tilde{\varphi}$. The capital S denotes an S-duality. For the details, see Sec. 3.

As for Tab. 1, note that the following duality chains including S-dualities are realized only if some parameter constraints are satisfied;

$$\begin{aligned} \text{S} \circ (\text{TsT})_\chi^{a_2} \circ (\text{TsT})_\varphi^{a_1} & \quad \text{if} \quad a_2 = \pm a_1, \\ (\text{TsT})_\varphi^{a_3} \circ \text{S} \circ (\text{TsT})_\chi^{a_2} \circ (\text{TsT})_\varphi^{a_1} & \quad \text{if} \quad a_2 = \pm a_1, \end{aligned}$$

³ The convention of the $\mathfrak{su}(4)$ algebra would be presented in [48]. The case with $b_1 = b_3 = 0$ will be elaborated in [48].

Classical r -matrices	Duality chains	Backgrounds
$r_{12}(a_1, 0, 0)$	$(\text{TsT})_\varphi^{a_1}$	(3.8), see also [45]
$r_{12}(a_1, a_2, 0)$	$(\text{TsT})_\chi^{a_2} \circ (\text{TsT})_\varphi^{a_1}$	(3.11)
$r_{12}(0, 0, a_1)$	$\text{So}(\text{TsT})_\chi^{\pm a_1} \circ (\text{TsT})_\varphi^{a_1}$	(3.13), see also [46]
$r_{12}(a_3, 0, a_1)$	$(\text{TsT})_\varphi^{a_3} \circ \text{So}(\text{TsT})_\chi^{\pm a_1} \circ (\text{TsT})_\varphi^{a_1}$	(3.17)
$r_{12}(a_3, a_4, a_1)$	$(\text{TsT})_\chi^{a_4} \circ (\text{TsT})_\varphi^{a_3} \circ \text{So}(\text{TsT})_\chi^{\pm a_1} \circ (\text{TsT})_\varphi^{a_1}$	(4.5)
$r_{12}(-a_1, -a_2, a_3)$	$\text{So}(\text{TsT})_\chi^{\pm a_3} \circ (\text{TsT})_\varphi^{a_3} \circ \text{So}(\text{TsT})_\chi^{a_2} \circ (\text{TsT})_\varphi^{a_1}$	(4.6)

Table 1: The classical r -matrix in (4.7) and the associated duality chains.

$$\begin{aligned}
& (\text{TsT})_\chi^{a_4} \circ (\text{TsT})_\varphi^{a_3} \circ \text{S} \circ (\text{TsT})_\chi^{a_2} \circ (\text{TsT})_\varphi^{a_1} & \text{if} & \quad a_2 = \pm a_1, \\
& \text{S} \circ (\text{TsT})_\chi^{a_4} \circ (\text{TsT})_\varphi^{a_3} \circ \text{S} \circ (\text{TsT})_\chi^{a_2} \circ (\text{TsT})_\varphi^{a_1} & \text{if} & \quad a_4 = \pm a_3.
\end{aligned}$$

In order to see the correspondence between the r -matrix in (4.7) and deformed backgrounds, we need to impose constraints for the values of a_1, a_2, a_3, a_4 . It seems likely that the constraints would come from the S-duality, but at the present moment we have no idea for the origin. To reveal it, it would be important to develop Yang-Baxter deformations of the D-string action. Then the origin may be understood as a consistency condition of the S-duality transformation.

5 Conclusion and discussion

We have further studied integrable deformations of the $\text{AdS}_5 \times \text{S}^5$ superstring based on the Yang-Baxter sigma model approach with classical r -matrices satisfying the CYBE. By focusing upon a classical r -matrix with two parameters and its three-parameter generalization, we have presented a family of the deformed metric and NS-NS two-form. The corresponding solutions of type IIB supergravity have been successfully obtained by performing TsT-transformations and S-dualities. In particular, it includes the solution found by Hubeny-Rangamani-Ross [45] and the one found by us [46] as special cases.

So far, we have seen that a certain multi-parameter family of classical r -matrices correspond to a chain of TsT-transformations and S-dualities for the undeformed $\text{AdS}_5 \times \text{S}^5$ background. It seems likely that Yang-Baxter sigma models based on CYBE could reproduce duality chains of $\text{AdS}_5 \times \text{S}^5$. In fact, three-parameter γ -deformations of S^5 [33, 34] and gravity duals for noncommutative gauge theories [36–39] have been reproduced from the associated classical r -matrices [35, 40]. Integrable deformations based on (at least) a cer-

tain class of r -matrices can be undone with non-local gauge transformations and twisted boundary condition, by following the philosophy of [52].

So far, we have discussed duality chains of $\text{AdS}_5 \times \text{S}^5$. This is a simple story and one may think of that Yang-Baxter sigma models would work only for duality chains. However, this is not the case. For example, more complicated backgrounds have been presented in [46] from the Yang-Baxter sigma model approach and those do not seem to be realized as TsT transformations of $\text{AdS}_5 \times \text{S}^5$ because of the singular structure. Therefore, it seems likely that the Yang-Baxter sigma model approach would include a wider class of gravity solutions which cannot be realized by performing string dualities for the undeformed $\text{AdS}_5 \times \text{S}^5$.

It would be of great importance to clarify the applicability of Yang-Baxter sigma models.

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Appendix

A Notation and convention

An explicit basis of $\mathfrak{su}(2, 2)$ is represented by the following γ -matrices,

$$\begin{aligned} \gamma_1 &= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \\ \gamma_0 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \gamma_5 = i\gamma_1\gamma_2\gamma_3\gamma_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \end{aligned} \quad (\text{A.1})$$

With the Lorentzian metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, the Clifford algebra is satisfied as

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \quad \{\gamma_\mu, \gamma_5\} = 0, \quad (\gamma_5)^2 = 1. \quad (\text{A.2})$$

The Lie algebra $\mathfrak{so}(1, 4)$ is formed by the generators

$$m_{\mu\nu} = \frac{1}{4} [\gamma_\mu, \gamma_\nu] , \quad m_{\mu 5} = \frac{1}{4} [\gamma_\mu, \gamma_5] \quad (\mu, \nu = 0, 1, 2, 3) , \quad (\text{A.3})$$

and then enlarged algebra $\mathfrak{so}(2, 4) = \mathfrak{su}(2, 2)$ is spanned by the following set:

$$m_{\mu\nu} , \quad m_{\mu 5} , \quad \frac{1}{2} \gamma_\mu , \quad \frac{1}{2} \gamma_5 . \quad (\text{A.4})$$

The reality condition for these generators are given by

$$M^\dagger \gamma_0 + \gamma_0 M = 0 \quad \text{for} \quad \forall M \in \mathfrak{su}(2, 2) . \quad (\text{A.5})$$

The generators p_μ used to parameterize $g \in SU(2, 2)$ in (2.9) are defined as

$$p_\mu \equiv \frac{1}{2} \gamma_\mu - m_{\mu 5} . \quad (\text{A.6})$$

To see the $\mathfrak{so}(2, 4)$ algebra explicitly, one can introduce the generators as follows:

$$\begin{aligned} \tilde{m}_{\mu\nu} &= \frac{1}{4} [\gamma_\mu, \gamma_\nu] , \quad \tilde{m}_{\mu 5} = \frac{1}{4} [\gamma_\mu, \gamma_5] \quad (\mu, \nu = 0, 1, 2, 3) , \\ \tilde{m}_{\mu, -1} &= -\tilde{m}_{-1, \mu} = \frac{1}{2} \gamma_\mu , \quad \tilde{m}_{5, -1} = -\tilde{m}_{-1, 5} = \frac{1}{2} \gamma_5 . \end{aligned} \quad (\text{A.7})$$

Indeed, these generators satisfy the commutation relations,

$$[\tilde{m}_{\hat{\mu}\hat{\nu}}, \tilde{m}_{\hat{\rho}\hat{\sigma}}] = \tilde{\eta}_{\hat{\rho}\hat{\nu}} \tilde{m}_{\hat{\mu}\hat{\sigma}} - \tilde{\eta}_{\hat{\rho}\hat{\mu}} \tilde{m}_{\hat{\nu}\hat{\sigma}} - \tilde{\eta}_{\hat{\sigma}\hat{\nu}} \tilde{m}_{\hat{\mu}\hat{\rho}} + \tilde{\eta}_{\hat{\sigma}\hat{\mu}} \tilde{m}_{\hat{\nu}\hat{\rho}} . \quad (\text{A.8})$$

Here $\hat{\mu}, \hat{\nu}, \hat{\rho}, \hat{\sigma} = -1, 0, 1, 2, 3, 5$ and $\tilde{\eta}_{\hat{\mu}\hat{\nu}} = \text{diag}(-1, -1, 1, 1, 1, 1)$.

B The T-duality rules

The rules of T-duality [53, 54] are summarized here. We basically follow Appendix C of [54].

The transformation rules between type IIB and type IIA supergravities are listed below. Note that the T-duality is performed for the y -direction and the other coordinates are denoted by a, b, a_i ($i = 1, \dots$). The fields of type IIB supergravity are the metric $g_{\mu\nu}$, NS-NS two-form B_2 , dilaton Φ , R-R gauge fields $C^{(2n)}$. The ones of type IIA supergravity are denoted with the tilde, the metric $\tilde{g}_{\mu\nu}$, NS-NS two-form \tilde{B}_2 , dilaton $\tilde{\Phi}$, and R-R gauge fields $\tilde{C}^{(2n+1)}$.

From type IIB to type IIA

$$\begin{aligned}
\tilde{g}_{yy} &= \frac{1}{g_{yy}}, & \tilde{g}_{ay} &= \frac{B_{ay}}{g_{yy}}, & \tilde{g}_{ab} &= g_{ab} - \frac{g_{ya}g_{yb} - B_{ya}B_{yb}}{g_{yy}}, \\
\tilde{B}_{ay} &= \frac{g_{ay}}{g_{yy}}, & \tilde{B}_{ab} &= B_{ab} - \frac{g_{ya}B_{yb} - B_{ya}g_{yb}}{g_{yy}}, & \tilde{\Phi} &= \Phi - \frac{1}{2} \ln g_{yy}, \\
\tilde{C}_{a_1 \dots a_{2n+1}}^{(2n+1)} &= -C_{a_1 \dots a_{2n+1}y}^{(2n+2)} - (2n+1)B_{y[a_1}C_{a_2 \dots a_{2n+1}]y}^{(2n)} + 2n(2n+1)\frac{B_{y[a_1}g_{a_2|y]}C_{a_3 \dots a_{2n+1}]y}^{(2n)}}{g_{yy}}, \\
\tilde{C}_{a_1 \dots a_{2n}y}^{(2n+1)} &= C_{a_1 \dots a_{2n}}^{(2n)} + 2n\frac{g_{y[a_1}C_{a_2 \dots a_{2n}]y}^{(2n)}}{g_{yy}}.
\end{aligned} \tag{B.1}$$

where the anti-symmetrization for indices is defined as, for example,

$$A_{[a}B_{b]} \equiv \frac{1}{2}(A_a B_b - A_b B_a).$$

The symbol $|y|$ inside the anti-symmetrization means that the indices other than the index y are anti-symmetrized.

From type IIA to type IIB

$$\begin{aligned}
g_{yy} &= \frac{1}{\tilde{g}_{yy}}, & g_{ay} &= \frac{\tilde{B}_{ay}}{\tilde{g}_{yy}}, & g_{ab} &= \tilde{g}_{ab} - \frac{\tilde{g}_{ya}\tilde{g}_{yb} - \tilde{B}_{ya}\tilde{B}_{yb}}{\tilde{g}_{yy}}, \\
B_{ay} &= \frac{\tilde{g}_{ay}}{\tilde{g}_{yy}}, & B_{ab} &= \tilde{B}_{ab} - \frac{\tilde{g}_{ya}\tilde{B}_{yb} - \tilde{B}_{ya}\tilde{g}_{yb}}{\tilde{g}_{yy}}, & \Phi &= \tilde{\Phi} - \frac{1}{2} \ln \tilde{g}_{yy}, \\
C_{a_1 \dots a_{2n}}^{(2n)} &= \tilde{C}_{a_1 \dots a_{2n}y}^{(2n+1)} - 2n\tilde{B}_{y[a_1}\tilde{C}_{a_2 \dots a_{2n}]y}^{(2n-1)} + 2n(2n-1)\frac{\tilde{B}_{y[a_1}\tilde{g}_{a_2|y]}\tilde{C}_{a_3 \dots a_{2n}]y}^{(2n-1)}}{\tilde{g}_{yy}}, \\
C_{a_1 \dots a_{2n-1}y}^{(2n)} &= -\tilde{C}_{a_1 \dots a_{2n-1}}^{(2n-1)} - (2n-1)\frac{\tilde{g}_{y[a_1}\tilde{C}_{a_2 \dots a_{2n-1}]y}^{(2n-1)}}{\tilde{g}_{yy}}.
\end{aligned} \tag{B.2}$$

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